Cycles in hamiltonian graphs of prescribed maximum degree

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Abstract

Let G be a hamiltonian graph G of order n and maximum degree Δ, and let $C(G)$ denote the set of cycle lengths occurring in G. It is easy to see that $|C(G)| \geq \Delta - 1$. In this paper, we prove that if $\Delta > \frac{n}{2}$, then $|C(G)| \geq \frac{n+\Delta-3}{2}$. We also show that for every $\Delta \geq 2$ there is a graph G of order $n \geq 2\Delta$ such that $|C(G)| = \Delta - 1$, and the lower bound in case $\Delta > \frac{n}{2}$ is best possible.

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1 Introduction

We consider only finite, undirected and simple graphs and we use Bondy and Murty's book [1] for terminology. In particular, for a graph G, we denote by $V = V(G)$ its vertex set and by $E = E(G)$ its set of edges. By C_p we mean a p-cycle of G, *i.e.* a cycle of length p. The vertices of a graph G of order n will be denoted by integers $1, 2, \ldots, n$. The edges of a complementary graph of a graph G are referred to as *red edges*. We denote by $C(G)$ the set of integers p, $3 \leq p \leq n$, such that G contains a cycle of length p.

The description of the set $S(n, \Delta)$ of cycle lengths occurring in every hamiltonian graph of order n and maximum degree Δ is given in [3] and [4]. Clearly, $S(n, \Delta) = \bigcap C(G)$, where the intersection is taken over all hamiltonian graphs of order n and maximum degree Δ. In particular, it is shown in [4] that a hamiltonian graph of order n and $\Delta > \frac{n}{2}$ contains a cycle C_p for every integer p belonging to the union

$$\bigcup_{s=1}^{n} \left(\frac{n-1-\Delta}{s} + 2, \frac{\Delta}{s} + 2\right).$$

This result was shown ([4]) to be best possible.

There are several results (see [2]-[7]) on the set of cycle lengths in a hamiltonian graph with given degree sum of two vertices.

The purpose of the present paper is the study of the number $|C(G)|$ of cycles lengths rather than the structure of $C(G)$. We give a lower bound of this number in dependence of the maximum degree Δ and the order of the graph. In particular we shall observe a "jump" of this bound in the neighborhood of the value $\Delta = \frac{n}{2}$. More precisely we shall prove the following theorem.

Theorem 1 *Let G be a hamiltonian graph of order n and maximum degree Δ. If $\Delta \leq \frac{n}{2}$, then $|C(G)| \geq \Delta - 1$. Moreover, for every $\Delta \geq 2$ there exist a graph G for which this bound is attained.*

If $\Delta > \frac{n}{2}$, then $|C(G)| \geq \frac{n}{2} + \frac{\Delta}{2} - \frac{3}{2}$. This bound is best possible.

The rest of the paper is organized as follows. The easy case of small values of Δ is considered below. Section 2 contains some lemmas. The last section is devoted to the second inequality of Theorem 1.

We shall use the following notation. The symbol G stands for a hamiltonian graph of order n with vertex set $[1, n] = \{1, 2, 3, \ldots, n-1, n\}$ and edge set E. By $C = (1, 2, 3, \ldots, n-1, n, 1)$ we denote a hamiltonian cycle of G. The degree of the vertex 1 is Δ, the maximum degree of G. The set of neighbors of 1 will be denoted by X. Note that with this notation, if $p \in X$ and $2 < p < n$, then $p \in C(G)$.

It is easily checked that for a hamiltonian graph G of maximum degree $\Delta \leq n/2$, $|C(G)| \geq \Delta - 1$. This is, in a sense, best possible in view of the construction below.

Let $k \geq 4$ and $q \geq 0$ be two integers. Define G as follows. The order of G equals $(q+2)(k-2)+2$ and the edge-set of G consists of the hamiltonian cycle $1, 2, 3, \ldots, n-1, n, 1$ and of the edges joining 1 with every vertex of the form $k + x(k-2)$, where $0 \leq x \leq q$. It is easy to see that G has maximum degree $\Delta = q + 3$ and that the cycles of G may have only $q + 1$ lengths of the form $k + x(k-2)$, $0 \leq x \leq q$ and, of course, one cycle of length n. Thus $|C(G)| = q + 2 = \Delta - 1$.

2 Some lemmas

For given $A \subset V$ we denote by $f(A)$ the number of neighbors of 1 in A *i.e.*

$$f(A) = |X \cap A|.$$

We start with some simple observations.

Proposition 2 *If $k \notin C(G)$, then $k \notin X$ and $n - k + 2 \notin X$.* ■

Proposition 3 *If $k \notin C(G)$ and $a \in X$ and $a+k-2 < n$, then $a+k-2 \notin X$.* ■

Corollary 4 *Let A and B be two disjoint subsets of $[1, n]$ with $B = A + (k-2)$. If $k \notin C(G)$ then*

$$f(A \cup B) = f(A) + f(B) \leq |A| = \frac{1}{2}(|A| + |B|).$$

Proof. The proof follows from the observation that if $x \in A \cap X$ then, by Proposition 3, the vertex $x + (k-2)$ belonging to B is not in X . ■

In particular, we shall use the last corollary when A and B are two consecutive segments (*i.e.* their union is also a segment) containing each $k-2$ elements. However, in this case we shall need a more general result.

Lemma 5 *Let $B_1, B_2, \ldots, B_{2t}$ be $2t$ disjoint, consecutive segments of $[1, n]$, each of length $k-2$. If $k \notin C(G)$, then*

$$f(\bigcup_{i=1}^{2t} B_i) \leq \frac{1}{2}\sum_{i=1}^{2t} |B_i|.$$

Proof. Since the number of segments B_i is $2t$, we can divide the segments into t pairs (B_1, B_2), (B_3, B_4),..., (B_{2t-1}, B_{2t}) and apply Corollary 4 to each pair separately. By adding the obtained inequalities we get the conclusion. ■

Since, for $t > 1$, B_1 and B_{2t} are not consecutive, the value of $f(B_1 \cup B_{2t})$ may be greater than $|B_1|$. However, in this case we have the following estimation.

Lemma 6 *With the same notation as in the previous lemma suppose that $k \notin C(G)$. Then*

$$f(B_1 \cup B_{2t}) \leq |B_1| + \xi,$$

where ξ is defined by

$$f(\bigcup_{i=2}^{2t-1} B_i) = \frac{1}{2}\sum_{i=2}^{2t-1} |B_i| - \xi.$$

Proof. The assertion of the lemma follows from Corollary 4 in the case $t = 1$. Therefore, suppose $t > 1$. Applying the previous lemma to the sequence of segments $B_2, \ldots, B_{2t-1}$ we deduce that $\xi \geq 0$. It suffices now to repeat the previous step to the sequence $B_1, \ldots, B_{2t}$. ■

The key lemma is the following:

Lemma 7 *If $|C(G)| < \frac{n}{2} + \frac{\Delta}{2} - \frac{3}{2}$, then there exists an integer p, $p \leq \frac{n+2}{2}$, such that*

$$p \notin C(G) \text{ and } n - p + 2 \notin C(G).$$

Proof. Let $C(G)^c = [3, n] \setminus C(G)$. We note that if $\frac{n+2}{2} \in C(G)^c$, then $\frac{n+2}{2}$ is the desired integer p. Assume that $\frac{n+2}{2} \notin C(G)^c$. If $k \in C(G)^c$, then Proposition 2 states that $k \notin X$ and $n-k+2 \notin X$. Now if there exist distinct $k, l \in C(G)^c$ such that $\{k, n-k+2\}$ intersects $\{l, n-l+2\}$, then $l = n-k+2 \in C(G)^c$, and k is the desired integer p. So, we suppose the sets $\{k, n-k+2\}$

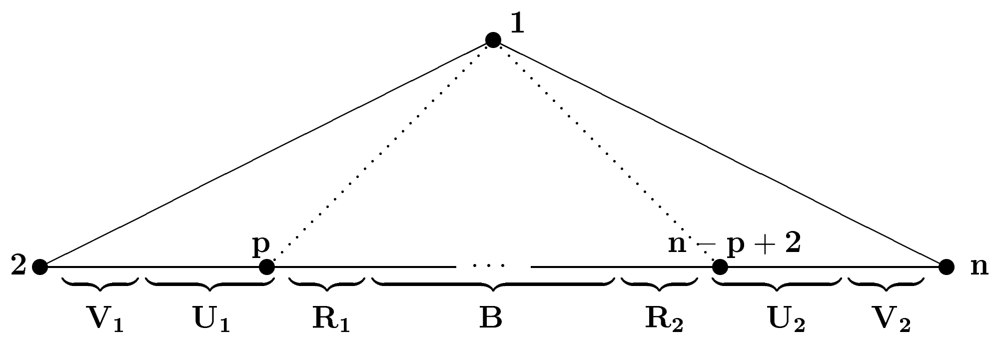

Figure 1:

with $k \in C(G)^c$ are pairwise disjoint. By Proposition 2, this implies that the number of red edges incident with vertex 1 is at least $2|C(G)^c|$. The degree of vertex 1 is then at most $n-1-2|C(G)^c| < n-1-(n-\Delta-1) = \Delta$, a contradiction. ■

3 Large values of Δ

Suppose, contrary to our claim, that there is a graph G of order n and $\Delta > \frac{n}{2}$ such that $|C(G)| < \frac{n}{2} + \frac{\Delta}{2} - \frac{3}{2}$. Let p, $3 \le p \le \frac{n+2}{2}$ be an integer satisfying the following property:

$$p \notin C(G) \text{ and } n-p+2 \notin C(G). \tag{$*$}$$

The existence of p is guaranteed by Lemma 7. If $p < \frac{n+2}{2}$ we have $n-2(p-2)-3 \ge 0$ vertices between p and $n-p+2$ on C. Let t and r be the quotient and the remainder when $n-2p+1$ is divided by $2(p-2)$, *i.e.*

$$n-2p+1 = 2t(p-2)+r \tag{$**$}$$

with $0 \le r < 2(p-2)$. If $p = \frac{n+2}{2}$ we put $t = r = 0$.

Let r_1, r_2 be two integers such that $r_1 + r_2 = r$, $0 \le r_1 \le r_2 \le r_1 + 1$. For $r_1 \ge 1$ we define two segments on C

$$R_1 = [p+1, p+r_1],$$

$$R_2 = [n-p-r_2+2, n-p+1].$$

In other words R_1 is the segment having r_1 vertices with first vertex $p+1$ and R_2 is the segment having r_2 vertices with last vertex $n-p+1$. For $r_1 = 0$ or $r_2 = 0$ the corresponding set R_i is, by definition, empty.

Denote by B the segment $[p+r_1+1, n-p-r_2+1]$. By the construction, the segment B consists of an even number of segments, each of lengths $(p-2)$.

We put $V_1 = \{i-(p-2) : i \in R_1\}$ and $V_2 = \{i+(p-2) : i \in R_2\}$. Hence $V_1 = [3, r_1+2]$ and $V_2 = [n-r_2, n-1]$. Of course, if R_i is empty, then the set V_i is empty too.

Finally denote by U_1, U_2 the remaining parts of the segments $[3,p]$ and $[n-p+2, n-1]$, respectively. In other words $U_1 = [r_1+3, p]$ and $U_2 = [n-p+2, n-r_2-1]$. Observe that the segments $U_1 \cup R_1$ and $U_2 \cup R_2$ are both of length $p-2$ (see Figure 1). By Lemma 5 we know that $f(B) \leq (1/2)|B|$.

Let us put

$$f(B) = (1/2)|B| - \xi \tag{1}$$

Applying Lemma 6 to the sequence of segments $U_1 \cup R_1, B, R_2 \cup U_2$ and using (1) we get

$$f(U_1 \cup R_1) + f(R_2 \cup U_2) \leq p - 2 + \xi \tag{2}$$

Applying Corollary 4 to the sets V_1 and R_1, as well as to the sets V_2 and R_2, we get

$$f(V_1) + f(R_1) \leq |R_1| = r_1, \tag{3}$$
$$f(V_2) + f(R_2) \leq |R_2| = r_2. \tag{4}$$

Consider the set $V_1 \cup U_1 \cup U_2 \cup V_2$. Suppose that there exists $x \in [3, p-1]$ with $x \in X$. Then $(n-p+x) \notin X$, for otherwise we would have a cycle of length $n-p+2$ defined by $1, x, x+1, \ldots, n-p+x, 1$, which contradicts the property $(*)$. By symmetry, we therefore obtain

$$f(V_1) + f(U_1) + f(U_2) + f(V_2) \leq p - 3. \tag{5}$$

Let us put $A = V_1 \cup U_1 \cup R_1 \cup R_2 \cup U_2 \cup V_2$. Observe that $|A| = 2(p-2) + r_1 + r_2$. Moreover, $n = 3 + |A| + |B|$. Adding the inequalities (2), (3), (4) and (5) we get

$$2f(A) \leq p - 2 + \xi + r_1 + r_2 + p - 3.$$

Hence
$$2f(A) \leq |A| + \xi - 1.$$

Thus
$$f(A) \leq \frac{|A|}{2} + \frac{\xi}{2} - \frac{1}{2}.$$

Using the last inequality, (1) and the fact that the edges $(1, 2)$ and $(1, n)$ are in E we get

$$\Delta = 2 + f(A) + f(B) \leq 2 + \frac{|A|}{2} + \frac{\xi}{2} - \frac{1}{2} + \frac{|B|}{2} - \xi \leq \frac{|A| + |B| + 3}{2} = \frac{n}{2},$$

a contradiction.

Finally, for given n define a graph G of order n and maximum degree Δ as follows: the edge-set of G consists of the hamiltonian cycle $1, 2, 3, \ldots, n - 1, n, 1$ and of the edges joining 1 with every vertex x where $\frac{n-\Delta+5}{2} \leq x \leq \frac{n+\Delta-1}{2}$ if $n - \Delta$ is odd and $\frac{n-\Delta+4}{2} \leq x \leq \frac{n+\Delta-2}{2}$ if $n - \Delta$ is even.

It is easy to see that G has indeed maximum degree Δ and that G have no cycle of length greater than $\frac{n+\Delta-1}{2}$ if $n - \Delta$ is odd and $\frac{n+\Delta}{2}$ if $n - \Delta$ is even (except for the cycle of length n). Thus $|C(G)| = \frac{n+\Delta-3}{2}$ if $n - \Delta$ is odd and $|C(G)| = \frac{n+\Delta-2}{2}$ if $n - \Delta$ is even. This finishes the proof of the theorem.

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